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Axioms for Nonrelativistic Quantum Mechanics

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Abstract

On the basis of the axioms assumed it is proved that the logic of propositions concerning any quantum-mechanical system may be endowed with the structure of an orthomodular atomistic complete lattice satisfying the covering postulate, and hence, as a consequence of these axioms, the Piron-MacLaren representation theorem for the logic is obtained.

1. *Introduction*

In our paper we come back to the old problem of giving a justification for the Hilbert-space description of quantum phenomena. Among many attempts to solve this problem the so-called "quantum-logic" approach seems to be very appropriate. Its main result is the well-known Piron-MacLaren representation theorem for the quantum logic (see Piron, 1964; MacLaren, 1964, 1965; also Varadarajan, 1968; Maeda and Maeda, 1970).

The axiom system presented here belongs to the "quantum logic" class and is interesting in the following two respects. Firstly, the complete lattice property of the logic is now a consequence of the axioms and not a postulate, and secondly, the covering postulate is formulated here in terms of atoms (pure states) only, and admits a simple interpretation.

2. *Axioms and Their Consequences*

The set of all experimentally verifiable propositions (questions, yes-no measurements) concerning a given physical system, called the *logic* of the system, we denote by L, and the set of states of the system by S.

The following two postulates are placed in a common origin of all "logical" approaches to quantum axiomatics and seem to be unquestionable:

> *Axiom 1.* L is an orthomodular σ-orthoposet.
> *Axiom 2.* The set S of states is a σ-convex subset of the set of all probability measures on L.

Also the following postulate seems to be natural (it was assumed, e.g., by Mackey, 1963):

> *Axiom 3.* For every nonzero proposition $a \in L$ there exists a state $m \in S$ with $m(a) = 1$.

The next axiom (compare Bugajska and Bugajski, 1972, Axiom 5) may be easily understood, if the partial ordering $\leqslant$ is interpreted as the implication between propositions.

> *Axiom 4.* If $m(a) = m(b) = 1$ for some $m \in S$ and $a, b \in L$, then there exists a proposition $c \leqslant a, b$ such that $m(c) = 1$.

We assume, additionally, the separability of the logic (see, for example, Zierler, 1961; Gunson, 1967):

> *Axiom 5.* The logic L is separable, that is, every subset of mutually orthogonal propositions from L is at most countable.

Definition. A proposition $a \in L$ is said to be the *carrier* of a given state $m \in S$ (see Zierler, 1961; Pool, 1968), if a is the smallest element in the set $\{b \in L: m(b) = 1\}$.

The carrier of m, whenever it exists, is obviously unique and we denote it by carr m.

One can show the following consequences of Axioms 1-5 (for proofs, see Bugajska and Bugajski, 1972):

2.1. Every state $m \in S$ has the carrier.

2.2. If $m = \Sigma_i t_i m_i$, where i runs over an at most countable set of indices, $t_i > 0$, $\Sigma_i t_i = 1$, then carr $m = \bigvee_i$ carr m_i.

2.3. For each nonzero proposition $a \in L$ there exists a state $m \in S$ such that $a =$ carr m.

2.4. The logic L is a lattice.

Moreover, one can easily show that

2.5. The logic L is a complete lattice.

Proof. Using 2.1-2.3 one can show (in exactly the same manner as the statement 2.4 was shown in the paper of Bugajska and Bugajski, 1972) that L is σ-complete. The completeness of L is then a consequence of the following fact (compare Varadarajan, 1968, pp. 183-184): Every separable σ-complete orthomodular lattice L is complete.

Let now $K \subseteq L$, $T \subseteq S$ and $j = 0$ or 1. The following abbreviations will be used throughout this paper:

$$K^j = \{m \in S: m(a) = j \quad \text{for all} \quad a \in K\}$$

$$T^j = \{a \in L: m(a) = j \quad \text{for all} \quad m \in T\}$$

If the set K consists of one point only, say $K = \{a\}$, then we write a^j instead of $\{a\}^j$. Analogously, we write m^j instead of $\{m\}^j$.

Definition. The proposition $a \in L$ is said to be the *carrier* of a given set $T \subseteq S$ $(T \neq \emptyset)$, if a is the smallest element in T^1.

From the definition above it follows easily that the carrier of T, whenever it exists, is unique; we denote it by carr T.

2.6. Let T be an arbitrary nonempty subset of S. Then there exists carr T, and carr $T = \bigvee_{m \in T}$ carr m.

Proof. To show that $\bigvee_{m \in T}$ carr m = carr T, suppose $a \in T^1$. Then $a \geqslant$ carr m for all $m \in T$, hence $a \geqslant \bigvee_{m \in T}$ carr m. Furthermore, $m(\bigvee_{m \in T} \text{carr } m) = 1$ for every $m \in T$ implies $\bigvee_{m \in T}$ carr $m \in T^1$, which completes the proof of the statement.

Let U and T be two nonempty subsets of S such that $U \subseteq T$. Denote

$$T(U) := \{m \in T: U^1 \subseteq m^1\}$$

2.7. Let $\emptyset \neq U \subseteq T \subseteq S$. Then $T(U) = (\text{carr } U)^1 \cap T$.

Proof. To prove the inclusion $(\text{carr } U)^1 \cap T \subseteq T(U)$ assume that $m \in (\text{carr } U)^1 \cap T$, and let $a \in U^1$ $(a \in L)$. Then obviously carr $U \leqslant a$, hence $m(a) \geqslant m(\text{carr } U) = 1$, which implies $a \in m^1$. Thus we have shown that $m \in T(U)$.

To show the inverse inclusion, let $m \in T(U)$. Then carr $U \in U^1 \subseteq m^1$, which implies $m \in (\text{carr } U)^1 \cap T$. This completes the proof of the statement.

Let T be a fixed but arbitrary nonempty subset of the set S of all states of the system. The following can then easily be verified:

2.8. The operation $U \to T(U)$ defined in the family of all nonempty subsets of T has the following properties:

(i) $U \subseteq T(U)$ for each $U \subseteq T$
(ii) $T(U_1) \subseteq T(U_2)$ whenever $U_1 \subseteq U_2$
(iii) $T(T(U)) = T(U)$ for every U.

In other words, the mapping $U \to T(U)$ is a kind of closure operation in the sense of Moore and Birkhoff, and therefore (see Birkhoff, 1948) the following holds:

2.9. Under set inclusion, the family $C(T)$ of all closed subsets of T [that is, such subsets of T, for which $U = T(U)$] becomes a complete lattice whose join and meet operations are given by

$$\bigvee_j U_j = T\left(\bigcup_j U_j\right) \quad \text{and} \quad \bigwedge_j U_j = \bigcap_j U_j$$

($\{U_j\}$ being an arbitrary family of closed subsets of the set T).

The family $C(T)$ may be completed by adjoining to it the empty set $\emptyset$. We then put, by the definition, $T(\emptyset) := \emptyset$.

Let now T be an arbitrary nonempty subset of S. The following can easily be shown:

2.10. For every $a \in L$ the set $a^1 \cap T$ is closed, and $a \leqslant b$ $(a, b \in L)$ implies $a^1 \cap T \subseteq b^1 \cap T$. Moreover, the map $a \to a^1 \cap T \in C(T)$ is a surjection.

Proof. One can assume without any loss of generality that $a^1 \cap T \neq \emptyset$. Let $m \in T(a^1 \cap T)$; since $a \in (a^1 \cap T)^1 \subseteq m^1$, we have $m \in a^1 \cap T$. This shows that $T(a^1 \cap T) = a^1 \cap T$.

The implication $a \leqslant b \Rightarrow a^1 \cap T \subseteq b^1 \cap T$ is obvious. By 2.7, for every non-empty $U \subseteq T$ with $T(U) = U$ one has $U = a^1 \cap T$, where $a = \text{carr } U$, which (together with $\emptyset = 0^1 \cap T$) proves that the map $a \to a^1 \cap T$ is a surjection.

2.11. $a^1 \subseteq b^1 (a, b \in L)$ implies $a \leqslant b$.

Proof. It can be assumed without any loss of generality that $a \neq 0$. Then, by 2.3, $a = \text{carr } m$ for some $m \in S$. But $m \in (\text{carr } m)^1 = a^1 \subseteq b^1$ implies $m(b) = 1$, hence $b \geqslant \text{carr } m = a$.

2.12. For every nonzero proposition $a \in L$ one has $a = \text{carr } a^1$.

Proof. As $a \in (a^1)^1$, it remains to be shown that $a \leqslant b$ for all $b \in (a^1)^1$. But $b \in (a^1)^1$ implies $a^1 \subseteq b^1$, hence $a \leqslant b$ by 2.11.

2.13. *Theorem.* Let T be a subset of S satisfying the following condition: (*) for each nonzero $a \in L$ the set $a^1 \cap T$ is nonempty. Then the mapping $a \to a^1 \cap T$ is an order isomorphism between L and $C(T)$; the inverse isomorphism [from $C(T)$ to L] is given by the mapping $U \to \text{carr } U$ (for $\emptyset$ we put, by the definition, carr $\emptyset := 0$).

The proof of Theorem 2.13 will be divided into a series of lemmas (Lemmas 2.14–2.16) below; T is always a fixed subset of S possessing the property (*).

2.14. *Lemma.* For every nonzero proposition $a \in L$ we have $a = \text{carr } (a^1 \cap T)$.

Proof. By 2.12 we get $a = \text{carr } a^1$. Since $a^1 \supseteq a^1 \cap T$, one has $a = \text{carr } a^1 \geqslant \text{carr } (a^1 \cap T)$. Suppose that $a > \text{carr } (a^1 \cap T)$. Then, by the orthomodularity of L, there exists $b \neq 0$ such that $b \perp \text{carr } (a^1 \cap T)$ and $a = b \vee \text{carr } (a^1 \cap T)$. By (*) there exists a state $m \in T$ with $m(b) = 1$, hence also $m(a) = 1$, that is $m \in a^1 \cap T$, which implies carr $m \leqslant \text{carr } (a^1 \cap T)$. Hence $m[\text{carr } (a^1 \cap T)] = 1$, which implies $m(a) = m(b) + m\,[\text{carr } (a^1 \cap T)] = 2$, a contradiction.

2.15. *Lemma.* For every set $W \in S$ including T as a subset one has $a^1 \cap W = W(a^1 \cap T)$.

Proof. One can assume without any loss of generality that $a \neq 0$. Then, by 2.7 and 2.14 one gets

$$W(a^1 \cap T) = [\text{carr } (a^1 \cap T)]^1 \cap W = a^1 \cap W$$

Putting $a = 1$ in 2.15 one obtains the following:

2.16. *Lemma.* If the conditions of the preceding lemma are satisfied, then $W = W(T)$, that is, T is "dense" in W.

2.17. *Lemma.* $a^1 \cap T \subseteq b^1 \cap T$ $(a, b \in L)$ implies $a \leqslant b$.

Proof. $a^1 \cap T \subseteq b^1 \cap T$ leads to $S(a^1 \cap T) \subseteq S(b^1 \cap T)$, hence (see 2.15) $a^1 \subseteq b^1$, which implies $a \leqslant b$ by 2.11.

Lemmas 2.14–2.17 prove our theorem 2.13.

Definition. We say that two states $m_1, m_2 \in S$ are *orthogonal* and write $m_1 \perp m_2$, if $m_1(a) = 1$ and $m_2(a) = 0$ for some proposition $a \in L$.

This orthogonality relation is obviously symmetric, that is $m_1 \perp m_2$ implies $m_2 \perp m_1$.

Now let T be a subset of S satisfying the condition (*) and let $\perp$ denotes the orthogonality in S restricted to the set T. For each $U \subseteq T$ define $U^\perp$ to be the set of all states $m \in T$ such that $m \perp U$ (read: $m \perp m_1$ for all $m_1 \in U$) and write U^- instead of $U^{\perp\perp}$. Obviously, $U \subseteq U^-$. If $U = U^-$, we call the set U *orthoclosed.* The family of all orthoclosed subsets of T will be denoted by $C(T, \perp)$; the map $U \to U^-$ $(U \subseteq T)$ is, obviously, a closure operation in the sense of Moore and Birkhoff, hence (see Birkhoff, 1948, Chap. IV, Theorem 1) under set inclusion $C(T, \perp)$ becomes a complete lattice with joins and meets defined by

$$\bigvee_j U_j = \left(\bigcup_j U_j\right)^- \qquad \text{and} \qquad \bigwedge_j U_j = \bigcap_j U_j$$

($\{U_j\}$ being an arbitrary family of orthoclosed subsets of T). Moreover, it can also easily be seen that $U \to U^\perp$ is an orthocomplementation in $C(T, \perp)$.

We shall now show that $C(T, \perp) = C(T)$. This is a consequence of the following statement:

2.18. For each subset $U \subseteq T$ one has $U^\perp = (\text{carr } U)'^1 \cap T$ and $U^- = T(U)$. (For the empty set $\emptyset$ we put by definition $\emptyset^\perp := T$).

Proof. Case I: $U^\perp = \emptyset$. We shall show that carr $U = 1$. Indeed, suppose that there exists $a \in U^1, a \neq 1$. Since $a' \neq 0$, there exists, by (*), a state $m \in S$ with $m(a') = 1$, hence $m(a) = 0$. Thus $m \in U^\perp$, which contradicts the assumption.

Case II. $U^\perp \neq \emptyset$. Let $m \in U^\perp$. Then carr $m \perp$ carr m_1 for all $m_1 \in U$, hence carr $m \perp \bigvee_{m_1 \in U}$ carr m_1 = carr U (see 2.6), hence $m(\text{carr } U) = 0$, that is $m \in (\text{carr } U)'^1 \cap T$.

Conversely, $m \in (\text{carr } U)'^1 \cap T$ implies $m(\bigvee_{m_1 \in U} \text{carr } m_1) = m(\text{carr } U) = 0$, hence $m(\text{carr } m_1) = 0$ for all $m_1 \in U$, which implies carr $m \perp$ carr m_1 for all $m_1 \in U$. Hence $m \perp m_1$ for every $m_1 \in U$, that is $m \in U^\perp$.

Thus the equality $U^\perp = (\text{carr } U)'^1 \cap T$ is proved. To show the second part of the statement note that owing to its first part we get

$$\begin{aligned} U^- &= (U^\perp)^\perp = (\text{carr } U^\perp)'^1 \cap T \\ &= [\text{carr}((\text{carr } U)'^1 \cap T)]'^1 \cap T \\ &= [(\text{carr } U)']'^1 \cap T \qquad \text{by 2.14} \\ &= (\text{carr } U)^1 \cap T \\ &= T(U) \qquad \text{by 2.7.} \end{aligned}$$

This completes the proof of the statement.

2.19. *Theorem.* If a subset $T \subseteq S$ satisfies the condition (*), then the order isomorphism between L and $C(T, \perp) = C(T)$ defined by

$$a \to a^1 \cap T, \qquad a \in L$$

is actually an orthoisomorphism. The orthoisomorphism inverse to the above one is defined by

$$M \to \text{carr } M, \qquad M \in C(T, \perp)$$

Proof. It would be shown that the mapping $a \to a^1 \cap T$ preserves the orthocomplementation. One can assume without any loss of generality that $a \neq 0$. By 2.14 and 2.18 one then gets

$$a'^1 \cap T = [\operatorname{carr}(a^1 \cap T)]'^1 \cap T = (a^1 \cap T)^\perp$$

which proves the theorem.

Since the set S of all states of the system satisfies, by the assumption (see Axiom 3), the property (*), as an immediated consequence of Theorem 2.19 one obtains the following:

2.20. *Theorem.* The mapping $a \to a^1, a \in L$, defines an orthoisomorphism of L onto $C(S, \perp)$; the orthoisomorphism inverse to the above one is defined by

$$M \to \operatorname{carr} M, \qquad M \in C(S, \perp)$$

Assume now, as an additional postulate, the following (see, e.g., Mac Laren, 1965):

> *Axiom 3'.* For every nonzero proposition $a \in L$ there exists a pure state p such that $p(a) = 1$.

Then, as a direct consequence of Theorem 2.19 we obtain the following:

2.21. *Theorem.* For any set $T \subseteq S$ including the set P of pure states as a subset the mapping

$$a \to a^1 \cap T, \qquad a \in L$$

defines an orthoisomorphism of L onto $C(T, \perp)$; the orthoisomorphism inverse to the above one is defined by

$$M \to \operatorname{carr} M, \qquad M \in C(T, \perp)$$

Remark. For particular cases, when $T = S$ or $T = P$, the theorems similar to 2.20 and 2.21 have been proved by Bugajska and Bugajski (1973a) under another axiom system.

> *Axiom 6.* For each pure state $p \in P$ one has $\{p\}^- = \{p\}$.

This axiom expresses the physically obvious fact that a single pure state cannot produce any superposition (see Bugajska and Bugajski, 1973a; Guz, 1974).

2.22. *Theorem.* The ortholattice $C(P, \perp)$ is atomistic and the mapping $p \to \operatorname{carr} p$, where $p \in P$, establishes a one-to-one correspondence between pure states and atoms of the logic L.

Proof. Atomisticity of $C(P, \perp)$ is a direct consequence of Axiom 6. The proof of the second part of the theorem is, in principle, the same as the proof of Lemma 2.9 of Guz (1974). We close our axiom system by formulating the following two axioms:

> *Axiom 7.* L is irreducible.
> *Axiom 8.* Given four atoms e_1, e_2, e_3, e such that $e \nleq e_1 \vee e_2$ and $e \leqslant e_1 \vee e_2 \vee e_3, e \neq e_3$, then there exists an atom f such that $f \leqslant e_1 \vee e_2$ and $f \leqslant e \vee e_3$.

Note that Axiom 7 is not restrictive. If it does not hold, then any irreducible part of the whole logic may be taken as L.

Axiom 8 is equivalent to the covering postulate (for the formulation of the latter see, for example, Mac Laren, 1964, or Maeda, 1970). The equivalence may be proved in an exactly the same way as Theorem (S) of Bugajska and Bugajski (1973b).

In order to understand the meaning of Axiom 8 two cases should be considered.

Case (a): $e_1 = e_2$. Then $f = e_1$ and Axiom 8 reduces to the following: $e \leqslant e_1 \vee e_3, e \neq e_1, e_3$ imply $e_1 \leqslant e \vee e_3$. This property seems to be obvious—see Figure 1.

Figure 1.

Case (b): $e_1 \neq e_2$. As $e \nleqslant e_1 \vee e_2$ and $e \leqslant e_1 \vee e_2 \vee e_3$, we have $e_3 \nleqslant e_1 \vee e_2$, and the situation that now appears is shown in Figure 2. In other words, two distinct lines have always a common point. (Remark: When the lines $e_1 \vee e_2$ and $e \vee e_3$ are parallel, f becomes the point at infinity.)

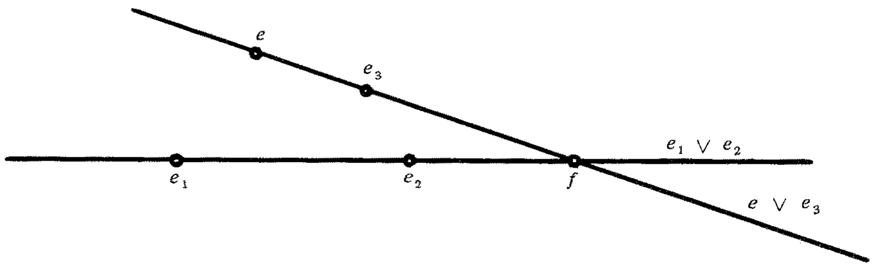

Figure 2.

3. *Conclusion*

To obtain the standard representation theorem for the logic L (see Piron, 1964; Mac Laren, 1965; Varadarajan, 1968; also Zierler, 1961) it now suffices to appeal to the following theorem (see Mac Laren, 1964; also Maeda, 1970):

Theorem. Let L be an irreducible atomistic complete orthocomplemented lattice of length $\geqslant 4$ with the covering property holding in it. Then there exists a division ring D with an involutive antiautomorphism $*$, a vector space V over D, and a definite (nondegenerate) $*$-bilinear form $(.\,,.)$ on V such that L is orthoisomorphic to the ortholattice of all $(.\,,.)$-closed linear manifolds in V.

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